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How Dark is the Shadow of a Round-Ended Screen?

JOSEPH B. KELLER

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Geometrical Theory of Elastic Surface-Wave Excitation and Propagation

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A geometrical theory is devised for the description and calculation of surface waves, such as Rayleigh waves, on boundaries or interfaces of elastic solids. It applies to curved surfaces and to inhomogeneous media. The theory, which is an extension of geometrical optics, involves complex rays that travel from the source to the surface, then along the surface, and finally from the surface to points in the solid. It also includes phases and amplitudes associated with each point on each ray. Geometrical formulas are derived for the determination of these phases and amplitudes. They involve certain excitation and radiation coefficients, which are also determined. The total displacement at a point is the sum of the displacements on all the rays through the point, each of which is constructed from the corresponding phase and amplitude. The theory applies to periodic waves of high frequency and short wavelength, as well as to the rapidly varying portions of any waveform. It is a generalization of the authors' earlier geometrical theory of scalar surface waves [J. Appl. Phys. 31, 1039 (1960)].

INTRODUCTION

A SURFACE wave is a wave that travels along a surface near which the wavemotion is confined, e.g., a Rayleigh wave. Because of the difficulty of solving exactly elasticity problems involving such waves, we have devised a geometrical method for their approximate solution. It is an extension of our theory of scalar surface waves¹ and is based upon geometrical optics. Therefore, it also provides a geometrical picture of how surface waves propagate. It is valid for periodic waves of high frequency and short wavelength and for the rapidly varying portions of any waveform. However, experience with other problems of wave propagation shows that it is accurate even for wavelengths as large as other dimensions in the problem.

Geometrical optics determines the rays along which waves travel. It also yields the phase or arrival time at each point, and can be used to find the amplitude and direction of the wavemotion. This leads to an expression for an elastic wavemotion that is the first term in an infinite series. This has been shown by V. M. Babich and A. S. Alekseev² and by us, and we have shown how to

determine the subsequent terms.³ However, although this series improves the description of the motion on the usual ray, it fails to describe diffracted and surface waves. An additional series is required for each such wave. We have shown how to construct the leading term in each series by an extension of geometrical optics, called the "geometrical theory of diffraction."⁴ That theory has been applied successfully to various scalar-, acoustic-, and electromagnetic-diffraction problems, and F. Gilbert⁵ has applied it to certain elastic-diffraction problems. Now, we show how to apply it to elastic surface waves.

In the next section, we introduce various new rays. In Sec. II, we determine the phase and amplitude of the wavemotion on each ray in terms of certain excitation and radiation coefficients. These coefficients are proportional to each other, as we show in Sec. III by means of the reciprocity principle. In Sec. IV, we apply the method to determine the Rayleigh wave produced on

³ F. C. Karal, Jr., and J. B. Keller, "Elastic Wave Propagation in Homogeneous and Inhomogeneous Media," J. Acoust. Soc. Am. 31, 694-705 (1959).

⁴ J. B. Keller, "A Geometrical Theory of Diffraction," in *Proceedings of Symposia in Applied Mathematics, Volume 8: Calculus of Variations and Its Applications*, edited by L. M. Graves (McGraw-Hill Book Co., Inc., New York, 1958).

⁵ F. Gilbert, "Scattering of Impulsive Elastic Waves by a Smooth Convex Cylinder," J. Acoust. Soc. Am. 32, 841-857 (1960).

¹ J. B. Keller and F. C. Karal, Jr., "Surface Wave Excitation and Propagation," J. Appl. Phys. 31, 1039-1046 (1960).

² V. M. Babich and A. S. Alekseev, "The Ray Method for Calculating the Intensity of Wave Fronts," Bull. Acad. Sci. USSR, Geophys. Ser. No. 1, 9-15 (1958).

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the plane surface on an elastic half-space by a compressional line source. By comparing the result with the known short-wavelength asymptotic form of the exact solution of the problem, we determine the excitation coefficient. This coefficient can then be used for any wave incident on any surface. All the preceding results apply to time-harmonic waves. In Sec. V, we transform them to apply to transient waves.

The theory, which can be applied easily to various problems, yields the first term in a series. Further terms can be derived by methods that have been used in similar scalar and electromagnetic problems.

I. COMPLEX RAYS

With each point \mathbf{x} of an isotropic elastic medium, there are associated two propagation velocities $V_c(\mathbf{x})$ and $V_s(\mathbf{x})$, called the compressional and shear velocities. In terms of each velocity, we can define rays in space, called compressional and shear rays, by means of Fermat's principle or the differential equations that follow from it. Each ray can be represented parametrically as a function $\mathbf{x}(s)$ of the arclength s . We also define complex rays of each type as complex valued functions $\mathbf{x}(s)$ that satisfy the ray differential equations or that are determined by Fermat's principle applied to complex curves. This definition is possible, provided the velocities are defined for complex values of \mathbf{x} , which is the case if they are analytic functions. Then, the complex rays are curves in a three-dimensional complex space. Of course, our primary interest in them concerns the points where they cross the real space. In unbounded media, the complex rays that start from a point in the real space and then leave this space usually never return to it. But, as we see here, in bounded media the complex rays may interact with the boundary and then return to the real space, in analogy with the fact that real rays can return to the vicinity of their source after reflection from a boundary.

Let us now consider a surface S along which a surface wave can propagate with the velocity $V_R(\mathbf{x})$. Real surface rays on S are curves that satisfy the differential equation for rays traveling on a surface with the velocity $V_R(\mathbf{x})$ or that are determined by the Fermat principle for curves on a surface. We define complex surface rays by applying Fermat's principle or the ray differential equations to complex curves on S . As before, this is possible if $V_R(\mathbf{x})$ and S are defined in the complex \mathbf{x} space.

When a space ray hits a surface S it gives rise to two reflected rays—one shear and one compressional—and two transmitted rays (provided there is a medium on the "transmitted" side of S). The directions of these rays are determined by the appropriate laws of reflection and refraction. These laws are consequences of Fermat's principle applied to curves having a point on the surface. It follows from them that a real incident ray produces two real reflected rays; but one or both of

the transmitted rays may be complex. The occurrence of complex transmitted rays is called total reflection and the waves associated with them are called evanescent waves.

If surface waves can propagate along S , we assume that the incident ray may also produce a surface ray. Then, the directions of the incident and surface rays are also related by the law of refraction in which the surface-ray velocity plays the role of the velocity along the transmitted ray. This is a consequence of applying Fermat's principle to curves having an arc on S on which the velocity is $V_R(\mathbf{x})$ and an arc in space. Since the surface ray lies in S , it follows from the law of refraction that the angle θ between the incident ray and the normal to S is given by

$$\sin\theta = V'(\mathbf{x})/V_R(\mathbf{x}). \quad (1)$$

Here, $V'(\mathbf{x})$ is either $V_c(\mathbf{x})$ or $V_s(\mathbf{x})$, according as the incident ray is a compressional or shear ray. This same relation holds when a space ray is shed by the surface ray, which we also assume to occur. In addition, the incident-ray direction, the surface-ray direction, and the surface normal must all lie in the same plane.

Surface waves are usually slow waves, which means that they travel slower than space waves and, consequently, $V'(\mathbf{x})/V_R(\mathbf{x}) > 1$ at real points \mathbf{x} . But then, by Eq. (1), θ cannot be real at real points, so, in this case, a real ray cannot produce a real surface ray. This is why we have to introduce complex rays to construct a ray description of surface-wave propagation.

There are certain surface waves that travel with the space velocity on the convex side of the surface. We have previously called them diffracted surface waves and the associated rays diffracted surface rays. In isotropic elastic media, there are two kinds: diffracted compressional and shear surface rays. When the incident ray and the diffracted surface ray are of the same type, then $V'(\mathbf{x}) = V_R(\mathbf{x})$ and it follows from Eq. (1) that $\theta = \pi/2$. Thus, an incident ray of either kind tangent to the surface can produce a diffracted surface ray of the same kind, and the latter can shed space rays of the same kind along its tangents. When the space ray and the diffracted surface ray are of different types, then it follows from Eq. (1) that the diffracted surface ray must have the greater velocity, as has been pointed out by Gilbert.⁵ We do not consider diffracted surface rays in the present work.

Let us now consider two points Q and P in an elastic medium bounded by a surface S along which a surface wave can propagate. There are direct and reflected rays from Q to P . In addition, a wave might travel from Q to P by going from Q to some point Q_1 on S , traveling along S from Q_1 to another point P_1 , and then leaving S at P_1 to go to P (see Fig. 1). We suppose that this happens, and we want to determine Q_1 and P_1 . The paths from Q to Q_1 and from P_1 to P are space rays, while that from Q_1 to P_1 is a surface ray. Since the rays QQ_1 and

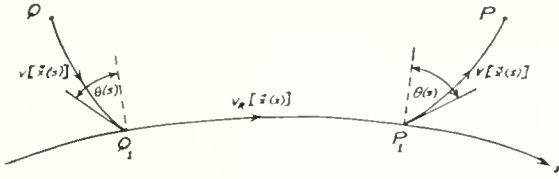


FIG. 1. The complex ray from Q reaches the surface at the complex point Q_1 where it produces a complex surface ray. This ray travels along the surface with the velocity $V_R[\mathbf{x}(s)]$ and sheds complex space rays. The ray that leaves the surface at P_1 passes through P . Since the rays QQ_1 and PP_1 may be either shear or compressional rays, there are four kinds of paths from Q to P and four pairs of points Q_1, P_1 . Each pair Q_1, P_1 is determined by the condition that Eq. (1) must be satisfied at each point with the appropriate value of $V[\mathbf{x}(s)]$.

PP_1 may be either shear or compressional rays, there are four kinds of paths from Q to P and four pairs of points Q_1, P_1 . Each pair Q_1, P_1 is determined by the condition that Eq. (1) must be satisfied at each point with the appropriate value of V .

II. PHASE AND AMPLITUDE

We now show how to use the rays to determine the elastic displacement $\mathbf{u}(P)$ at the point P . We first postulate that there is a displacement associated with each ray through P , and that the total displacement at P is the sum of the displacements on all the rays through P . We now assume that the motion is time-harmonic with angular frequency ω , and we write the displacement as the real part of $u(P)e^{-i\omega t}$, letting u be complex. Then, on each ray we write $u(P) = A(P)e^{i\varphi(P)}$, and we call $\varphi(P)$ the phase at P and $A(P)$ the amplitude at P . This representation of u is made unique by the requirement that $\varphi(P)$ be proportional to ω , while the amplitude A may be complex but may not contain any phase factor proportional to ω . Finally, we assume that $\varphi(P)$ is given by the relation

$$\varphi(P) = \varphi(P_1) + \omega \int_{P_1}^P ds/V[\mathbf{x}(s)]. \quad (2)$$

The integral in Eq. (2) is with respect to arclength s along the ray from P_1 to P and V is the velocity associated with the ray, which is either V_c , V_s , or V_R , depending upon the type of ray.

Let us apply Eq. (2) to each of the four paths from Q to P that contain arcs on S . We may write all four results together by introducing the subscripts i and j , each of which can take the two values c and s , denoting compressional and shear rays. Then, $\varphi_{ij}(P)$ denotes the phase at P on a ray whose first part, from Q to Q_1 , is of type i and whose last part, from P_1 to P , is of type j . The values of Q_1 and P_1 on such a ray are denoted by Q_{ij} and P_{ij} . Then, we have

$$\begin{aligned} \varphi_{ij}(P) = & \varphi_i(Q) + \omega \int_Q^{Q_{ij}} ds/V_i[\mathbf{x}(s)] \\ & + \omega \int_{Q_{ij}}^{P_{ij}} ds/V_R[\mathbf{x}(s)] + \omega \int_{P_{ij}}^P ds/V_j[\mathbf{x}(s)]. \end{aligned} \quad (3)$$

Here, $\varphi_i(Q)$ denotes the phase at Q on a ray of type i .

To determine the variation along a ray of the amplitude $A(P) = |\mathbf{A}(P)|$, we apply the principle of energy conservation to a tube of rays. For a space ray of type i , this principle states that $A^2 \rho V_i d\tau$ is constant along the tube where ρ is the density of the medium and $d\tau$ is the cross-sectional area of the tube. This follows from Eqs. (115) and (139) of Ref. 3 for compressional rays and from Eq. (113) for shear rays. We assume that the same principle applies to a strip of surface rays in the form $A^2 \rho' V_R d\sigma = \text{const}$, where $d\sigma$ denotes the width of the strip and ρ' is a density, depending upon the densities on the two sides of the surface.

To find the direction of $\mathbf{A}(P)$, we utilize the facts that on a compressional ray \mathbf{A} is tangential and on a shear ray \mathbf{A} is normal. Thus, in the compressional case, we write $\mathbf{A}_c = A_c \mathbf{k}$, where \mathbf{k} is a unit vector tangent to the ray, while in the shear case we write $\mathbf{A}_s = A_s \mathbf{n}$, where \mathbf{n} is the appropriate unit vector normal to the ray. Then, utilizing the energy principle that we have on the compressional and shear rays, respectively,

$$A_c(P) = A_c(P_1) \left[\frac{d\tau(P_1) \rho(P_1) V_c(P_1)}{d\tau(P) \rho(P) V_c(P)} \right]^{\frac{1}{2}} \mathbf{k}(P), \quad (4)$$

$$A_s(P) = A_s(P_1) \left[\frac{d\tau(P_1) \rho(P_1) V_s(P_1)}{d\tau(P) \rho(P) V_s(P)} \right]^{\frac{1}{2}} \mathbf{n}(P). \quad (5)$$

The variation of $\mathbf{n}(P)$ along the ray depends upon the torsion of the ray¹ and for a straight ray $\mathbf{n}(P)$ is constant. The direction of the amplitude associated with a surface ray is completely determined by the direction of the ray and the properties of the media at the surface and, therefore, we need not consider this direction. Thus, letting \hat{A} denote the amplitude on a surface ray, we write merely

$$\hat{A}(P) = \hat{A}(P_1) \left[\frac{d\sigma(P_1) \rho'(P_1) V_R(P_1)}{d\sigma(P) \rho'(P) V_R(P)} \right]^{\frac{1}{2}}. \quad (6)$$

Let us now determine the amplitudes $\mathbf{A}_{ij}(P)$ on each of the four rays that go from Q to P by way of the surface S . To do so, we make use of Eqs. (4)–(6) and certain additional relations between the amplitudes on the surface and space rays at the point on the surface where they meet. Let $\mathbf{A}_c(Q_{cj})$ and $\mathbf{A}_s(Q_{sj})$ be the amplitudes on the incident compressional and shear rays, respectively, at the points Q_{cj} and Q_{sj} . Then, we assume that the amplitudes $\hat{A}_c(Q_{cj})$ and $\hat{A}_s(Q_{sj})$ on the resulting surface rays are proportional to the incident amplitudes. Thus, we write,

$$\hat{A}_c(Q_{cj}) = E_c(Q_{cj}) A_c(Q_{cj}), \quad (7)$$

$$\hat{A}_s(Q_{sj}) = E_s(Q_{sj}) A_s(Q_{sj}) \cdot \mathbf{N}(Q_{sj}). \quad (8)$$

In Eqs. (7) and (8), $E_c(Q_{cj})$ and $E_s(Q_{sj})$ are excitation coefficients that, we assume, depend only upon the properties of the media at Q_{cj} , respectively. In Eq. (8),

$\mathbf{n}(Q_{sj})$ denotes the unit normal to S at Q_{sj} . We have introduced it because only the component of displacement in the plane of incidence is effective in exciting the surface wave, and this is proportional to the normal component.

We next assume that the amplitude $\mathbf{A}_{ij}(P_{ij})$ on a ray leaving the surface is proportional to the amplitude $\hat{A}_i(P_{ij})$ on the surface ray. We express this proportionality in terms of the radiation coefficients $R_j(P_{ij})$, which depend upon the properties of the media at P_{ij} . Thus, we write

$$A_{ij}(P_{ij}) = R_j(P_{ij})\hat{A}_i(P_{ij}). \quad (9)$$

When the final ray is a shear ray, $j=s$, we assume that the unit vector $\mathbf{n}(P_{is})$, which determines the direction of the displacement at P_{is} , lies in the plane containing the tangents to the surface ray and the shear ray at P_{is} . It is, of course, also normal to the shear ray.

Upon combining Eqs. (4)–(9), we obtain

$$A_{ij}(P) = \left[\frac{d\tau(P_{ij})}{d\tau(P)} \frac{\rho(P_{ij})}{\rho(P)} \frac{V_j(P_{ij})}{V_j(P)} \right]^{\frac{1}{2}} \times R_j(P_{ij}) \left[\frac{d\sigma(Q_{ij})}{d\sigma(P_{ij})} \frac{\rho'(Q_{ij})}{\rho'(P_{ij})} \frac{V_R(Q_{ij})}{V_R(P_{ij})} \right]^{\frac{1}{2}} \times E_i(Q_{sj})\hat{A}_i(Q_{ij}). \quad (10)$$

Let us also write the phase $\varphi_{ij}(P)$, given by Eq. (3), in the form

$$\varphi_{ij}(P) = \varphi_i(Q_{ij}) + \omega \int_{Q_{ij}}^{P_{ij}} ds/V_R[x(s)] + \omega \int_{P_{ij}}^P ds/V_j[x(s)]. \quad (11)$$

Then, we may write the total displacement at P due to surface rays in the form

$$\mathbf{u}(P) = A_{cc}(P)\mathbf{k}_{cc}(P)e^{i\varphi_{cc}(P)} + A_{sc}(P)\mathbf{k}_{sc}(P)e^{i\varphi_{sc}(P)} + A_{cs}(P)\mathbf{n}_{cs}(P)e^{i\varphi_{cs}(P)} + A_{ss}(P)\mathbf{n}_{ss}(P)e^{i\varphi_{ss}(P)}. \quad (12)$$

Here, $\mathbf{k}_{cc}(P)$ and $\mathbf{k}_{sc}(P)$ are unit tangents to the compressional rays at P , while $\mathbf{n}_{cs}(P)$ and $\mathbf{n}_{ss}(P)$ are the appropriate unit normals to the shear rays at P .

Equation (12), together with (10) and (11), expresses the main result of our geometrical theory of elastic surface waves. It shows how to express the elastic displacement at P due to such waves as a sum of terms. There is one term for each ray through P that comes from a surface ray on S . In Eq. (12), we have written four terms, since usually there will be four rays through P if the source of the motion is a point. For some geometries and for other sources, there may be more rays and, therefore, more terms must be included in Eq. (12). Each term involves the amplitude and phase on the incident ray at the point where it produces the surface

ray. It also involves the surface-ray velocity V_R , the density ρ' , the two excitation coefficients E_c and E_s , and the two radiation coefficients R_c and R_s . These quantities must be determined by other considerations—for example, from the solutions of special or canonical problems. Once they are known, the computation of $\mathbf{u}(P)$ is reduced to a geometrical calculation.

In the next section, we show that the excitation coefficients are expressible in terms of the radiation coefficients. In the subsequent sections, we determine these coefficients, as well as V_R and ρ' , for a free surface by using the solution of a canonical problem. The same method can also be used for any other type of surface.

III. RECIPROCITY PRINCIPLE

There exists a relation between the excitation and radiation coefficients, as we now show by utilizing the reciprocity principle.⁶ This principle may be formulated as follows:

$$u_x(P, Qy) = u_y(Q, Px). \quad (13)$$

Here, $u_x(P, Qy)$ denotes the x component of displacement at P due to a unit force in the y direction applied to the medium at Q . Similarly, $u_y(Q, Px)$ is the y component of displacement at Q due to a unit force in the x direction at P . We apply Eq. (13) to that part of the displacement \mathbf{u} that involves surface waves, which we have determined in the preceding section.

It is simplest to apply Eq. (13) to a two-dimensional problem in which the media, the surface, and \mathbf{u} are all independent of one coordinate, say z . The results will then apply only to such two-dimensional configurations. However, we assume that the excitation and radiation coefficients depend only upon properties of the surface and the media in the plane of incidence, at the points of excitation and radiation, respectively. Then, the two-dimensional results concerning these coefficients will apply in general.

On the basis of these considerations, we consider the displacement \mathbf{u} produced by a unit force in the y direction at Q . (In three dimensions, this point force represents a line source through Q parallel to the z axis.) Then, the phase $\varphi_i(Q_{ij})$ can be expressed by Eq. (2) in the form

$$\varphi_i(Q_{ij}) = \omega \int_Q^{Q_{ij}} ds/V_i[x(s)]. \quad (14)$$

The amplitudes $\mathbf{A}_c(Q_{cj})$ and $\mathbf{A}_s(Q_{sj})$ can be expressed in the form Eqs. (4) and (5). However, since all the rays lie in planes $z = \text{const}$, the areas $d\tau$ in these and the subsequent equations may be taken to be lengths in the xy plane multiplied by a unit length in the z direction. For the same reason, the lengths $d\sigma$ may be taken to

⁶ Lord Rayleigh, *The Theory of Sound* (Macmillan and Co. Ltd., London, 1894; reprinted, Dover Publications, Inc., New York, 1945), Vol. 1, p. 153–155.

be constant lengths in the z direction, so that $d\sigma(Q_{ij})/d\sigma(P_{ij})=1$.

In applying Eqs. (4) and (5), we replace P by Q and choose P_1 to be a point near Q . But then, $d\tau(P_1)=s d\theta(Q)$, where s is the distance from Q to P_1 and $d\theta(Q)$ is the angle between the two rays bounding a tube. As P_1 tends to Q , s tends to zero and $A_i(P_1)$ becomes infinite in such a way that the following limit exists: $\lim_{s \rightarrow 0} s^{1/2} A_i(P_1) = A_i'(Q)$. For a line source consisting of a unit force per unit length in the x direction in an infinite homogeneous medium, $A_i'(Q)$ is given by the following expression,⁷ in which we write $A_i^x(x)$ to indicate the direction of the force:

$$A_c^x(Q) = e^{i\pi/4} (8\pi\omega)^{-1/2} [V_c(Q)]^{-1/2} [\rho(Q)]^{-1} \sin\theta(Q), \quad (15)$$

$$A_s^x(Q) = e^{i\pi/4} (8\pi\omega)^{-1/2} [V_s(Q)]^{-1/2} [\rho(Q)]^{-1} \cos\theta(Q). \quad (16)$$

In Eqs. (15) and (16), $\theta(Q)$ is the angle that the ray makes with the positive y axis at Q . The unit vector \mathbf{n} is oriented so that $\mathbf{k} \times \mathbf{n}$ points in the positive z direction. In accordance with the geometrical method, we assume that $A_i^x(Q)$ is also given by Eqs. (15) or (16) for inhomogeneous and bounded media. Now, Eqs. (4) and (5) yield with P_1 and P replaced by Q and Q_{ij} , respectively,

$$A_i^x(Q_{ij}) = e^{i\pi/4} (8\pi\omega)^{-1/2} [V_i(Q)]^{-1/2} \times \left[\frac{d\theta(Q)}{d\tau(Q_{ij})} \frac{1}{\rho(Q)\rho(Q_{ij})V_i(Q_{ij})} \right]^{1/2} \frac{\sin\theta_{ij}(Q)}{\cos\theta_{ij}(Q)}. \quad (17)$$

Here, and in subsequent equations, \sin is to be used for $i=c$ and \cos for $i=s$.

Upon inserting Eq. (17) into Eq. (10), we obtain

$$A_{ij}^x(P) = e^{i\pi/4} (8\pi\omega)^{-1/2} \left[\frac{d\tau(P_{ij})}{d\tau(P)} \frac{\rho(P_{ij})}{\rho(P)} \frac{V_j(P_{ij})}{V_j(P)} \times \frac{\rho'(Q_{ij})}{\rho'(P_{ij})} \frac{V_R(Q_{ij})}{V_R(P_{ij})} \frac{d\theta(Q)}{d\tau(Q_{ij})} \frac{1}{\rho(Q)\rho(Q_{ij})V_i(Q_{ij})} \right]^{1/2} \times [V_i(Q)]^{-1/2} R_j(P_{ij}) E_i(Q_{ij}) \frac{\sin\theta_{ij}(Q)}{\cos\theta_{ij}(Q)}. \quad (18)$$

Inserting Eq. (14) into Eq. (11) yields

$$\varphi_{ij}(P) = \omega \int_Q^{Q_{ij}} \frac{ds}{V_i} + \omega \int_{Q_{ij}}^{P_{ij}} \frac{ds}{V_R} + \omega \int_{P_{ij}}^P \frac{ds}{V_j}. \quad (19)$$

When Eqs. (18) and (19) are used in Eq. (12), they yield $\mathbf{u}(P)$. The y component of $\mathbf{u}(P)$ is

$$u_y(P) = A_{cc}(P) \cos\theta_{cc}(P) e^{i\varphi_{cc}(P)} + A_{sc}(P) \cos\theta_{sc}(P) e^{i\varphi_{sc}(P)} + A_{cs}(P) \sin\theta_{cs}(P) e^{i\varphi_{cs}(P)} + A_{ss}(P) \sin\theta_{ss}(P) e^{i\varphi_{ss}(P)}. \quad (20)$$

Here, θ_{ij} is the angle between \mathbf{k}_{ij} and the y axis, and we have utilized the fact that \mathbf{n}_{is} lies in the xy plane and is perpendicular to \mathbf{k}_{is} . Similarly, we find

$$u_x(Q) = A_{cc}(Q) [-\sin\theta_{cc}(Q)] e^{i\varphi_{cc}(Q)} + A_{sc}(Q) [-\sin\theta_{sc}(Q)] e^{i\varphi_{sc}(Q)} + A_{cs}(Q) [\cos\theta_{cs}(Q)] e^{i\varphi_{cs}(Q)} + A_{ss}(Q) [\cos\theta_{ss}(Q)] e^{i\varphi_{ss}(Q)}. \quad (21)$$

To obtain $u_x(Q)$, we use Eqs. (18) and (19) with P and Q interchanged, taking account of the fact that the rays from Q to P are also the rays from P to Q . We still let i denote the type of ray from Q_{ij} to Q and j denote the type of ray from P to P_{ij} so that the labeling of the points P_{ij} and Q_{ij} is unchanged. Then, we find from Eq. (19) that $\varphi_{ij}(P) = \varphi_{ji}(Q)$. Since $u_x(Q)$ results from a force in the y direction at P , the coefficient $A_i^y(P)$ is obtained from Eqs. (15) and (16) by replacing θ by $\theta - (\pi/2)$.

We now insert Eqs. (20) and (21) into Eq. (13). The resulting equation must hold for all points P and Q and all values of ω . Therefore, the coefficients of $e^{i\varphi_{ij}(Q)}$ and $e^{i\varphi_{ji}(P)}$ must be equal. This yields the four equations

$$A_{cc}(P) \cos\theta_{cc}(P) = -A_{cc}(Q) \sin\theta_{cc}(Q); \quad (22)$$

$$A_{sc}(P) \cos\theta_{sc}(P) = A_{cs}(Q) \cos\theta_{cs}(Q); \quad (23)$$

$$A_{cs}(P) \sin\theta_{cs}(P) = -A_{sc}(Q) \sin\theta_{sc}(Q); \quad (24)$$

$$A_{ss}(P) \sin\theta_{ss}(P) = A_{ss}(Q) \cos\theta_{ss}(Q). \quad (25)$$

In these equations, we now use Eq. (18) and the corresponding expression with P and Q interchanged, noting that, in interchanging P and Q , $\theta_{ij}(Q)$ must be replaced by $\theta_{ij}(P) - (\pi/2)$. We then find that the trigonometric factors and the minus signs in these equations cancel out and that they all become of the form $A_{ij}(P) = A_{ji}(Q)$, with the sines and cosines omitted. We now use Eq. (18) for $A_{ij}(P)$ and for $A_{ji}(Q)$, equate these two expressions omitting the trigonometric factors, and transpose all factors involving P and P_{ij} to one side and those involving Q and Q_{ij} to the other. After simplifying, we can write the resulting equation in the form

$$\left[\left\{ \frac{d\tau(P_{ij})}{d\tau(P)} \right\}_Q \left\{ \frac{d\tau(P_{ij})}{d\theta(P)} \right\}_P \right]^{1/2} \times \frac{V_j(P_{ij})\rho(P_{ij})[V_j(P)]^{1/2} R_j(P_{ij})}{V_R(P_{ij})\rho'(P_{ij}) E_j(P_{ij})} = \left[\left\{ \frac{d\tau(Q_{ij})}{d\tau(Q)} \right\}_P \left\{ \frac{d\tau(Q_{ij})}{d\theta(Q)} \right\}_Q \right]^{1/2} \times \frac{V_i(Q_{ij})\rho(Q_{ij})[V_i(Q)]^{1/2} R_i(Q_{ij})}{V_R(Q_{ij})\rho'(Q_{ij}) E_i(Q_{ij})}. \quad (26)$$

In Eq. (26), the subscript Q or P on a geometrical ratio

⁷ H. Lamb, "On the Propagation of Tremors over the Surface of an Elastic Solid," Phil. Trans. Royal Soc. (London) **203**, 1-42 (1904).

indicates the source of the rays to be used in calculating that ratio.

The geometrical ratios in Eq. (26) can be calculated simply by letting Q approach Q_{ij} so that the ray from Q to Q_{ij} is essentially straight, and similarly for P and P_{ij} . Then, $\{d\tau(Q_{ij})/d\theta(Q)\}_{Q=Q_{ij}}$ is the distance from Q_{ij} to Q , which will be complex if Q_{ij} is complex. Furthermore, from Ref. 1, Eq. (10), we find when the rays are straight that

$$\left\{ \frac{d\tau(Q_{ij})}{d\tau(Q)} \right\}_P = [1 + (\kappa + \dot{\theta}_i) d(Q_{ij}, Q) \sec \theta_i]_{Q_{ij}}^{-1}. \quad (27)$$

Here, κ denotes the curvature of the surface at Q_{ij} , θ_i is the value of θ given by Eq. (1) at Q_{ij} with $V = V_i(Q_{ij})$, and $\dot{\theta}_i$ is the derivative of θ_i with respect to arclength along the surface. From Eq. (27) and the preceding ratio, we have

$$\left\{ \frac{d\tau(Q_{ij})}{d\tau(Q)} \right\}_P \left\{ \frac{d\tau(Q_{ij})}{d\theta(Q)} \right\}_Q = [\kappa_0 + (\kappa + \dot{\theta}_i) \sec \theta_i]_{Q_{ij}}^{-1}. \quad (28)$$

Here, $\kappa_0 = 1/d(Q_{ij}, Q)$ is the curvature of the wavefront incident at Q_{ij} .

We now use Eq. (28) on the right side of Eq. (26), and on the left side we insert the corresponding expression with P 's and Q 's interchanged and with θ_j in place of θ_i . Then, the left side of Eq. (26) is found to be independent of Q and the right side independent of P . Therefore, each side is a constant, which we shall write as a^{-2} . Furthermore, since we have let P approach P_{ij} , we may replace P by P_{ij} on the left side and similarly Q by Q_{ij} on the right. Upon equating the left side of Eq. (26) to a^{-2} , we obtain an equation for $E_j(P_{ij})$ that yields

$$E_j(P_{ij}) = a^2 [\kappa_0 + (\kappa + \dot{\theta}_j) \sec \theta_j]_{P_{ij}}^{-1} [V_j(P_{ij})]^{\frac{1}{2}} \times R_j(P_{ij}) \rho(P_{ij}) / V_R(P_{ij}) \rho'(P_{ij}). \quad (29)$$

It is convenient to introduce the new radiation coefficient $R_j'(P_{ij})$ defined by

$$R_j'(P_{ij}) = a R_j(P_{ij}) [V_j(P_{ij})]^{\frac{1}{2}} \times [\rho'(P_{ij}) V_R(P_{ij})]^{-1} [\rho(P_{ij})]^{\frac{1}{2}}. \quad (30)$$

Then, Eq. (29) becomes

$$E_j(P_{ij}) = a [\kappa_0 + (\kappa + \dot{\theta}_j) \sec \theta_j]_{P_{ij}}^{-1} \times \frac{[V_j(P_{ij})]^{\frac{1}{2}} [\rho(P_{ij})]^{\frac{1}{2}}}{[V_R(P_{ij})]^{\frac{1}{2}} [\rho'(P_{ij})]^{\frac{1}{2}}}. \quad (31)$$

This is the relationship between E_j and R_j that we have been seeking. We see from Eq. (26) that it also holds when P_{ij} is replaced by Q_{ij} .

Although Eq. (31) has been derived in the special case of cylindrical symmetry (i.e., for the two-dimensional case), we assume that it holds in general, provided that κ_0 and κ are the curvatures of the incident wave-

front and the surface in the plane of incidence. This is the plane containing the tangent to the incident ray and the normal to the surface. In addition, $\dot{\theta}_j$ is the derivative of θ_j along the surface ray.

Let us now use Eq. (31) for $E_i(Q_{ij})$ and eliminate E_i from Eq. (10). In this way, we obtain

$$A_{ij}(P) = A_i(Q_{ij}) R_j'(P_{ij}) R_i(Q_{ij}) \times \left[\frac{V_i(Q_{ij})}{V_j(P_{ij})} \right]^{\frac{1}{2}} \left[\frac{V_i(Q_{ij}) \rho(Q_{ij})}{V_j(P) \rho(P)} \frac{d\sigma(Q_{ij})}{d\sigma(P_{ij})} \frac{d\tau(P_{ij})}{d\tau(P)} \right]^{\frac{1}{2}} \cdot [\kappa_0 + (\kappa + \dot{\theta}_i) \sec \theta_i]_{Q_{ij}}^{-\frac{1}{2}}. \quad (32)$$

When Eq. (32) is used in Eq. (12), it yields the surface-wave field at P due to an arbitrary wave incident upon the surface. We note that Eq. (32) does not involve ρ' , which has been absorbed into the radiation coefficient R_i' . In the next section, we determine R_i' for a free surface, using a method that can also be applied to other kinds of surfaces. Once R_i' is known, Eqs. (12) and (32) will express the surface-wave field in terms of known geometrical quantities and material properties.

For a homogeneous medium bounded by a plane surface, the space rays and surface rays are straight lines. Then, Eq. (3) yields, with $\varphi_i(Q) = 0$,

$$\varphi_{ij}(P) = \omega d(Q, Q_{ij}) / V_i + \omega d(Q_{ij}, P_{ij}) / V_R + \omega d(P_{ij}, P) / V_j. \quad (33)$$

In the two-dimensional case of a homogeneous medium, Eq. (32) becomes

$$A_{ij}(P) = R_j' R_i' (V_i / V_j)^{\frac{1}{2}} [\kappa_0(Q_{ij})]^{-\frac{1}{2}} A_i(Q_{ij}). \quad (34)$$

In Eq. (34), we have made use of the facts that $\dot{\theta}_i = 0$, $\kappa = 0$, $d\sigma(Q_{ij})/d\sigma(P_{ij}) = 1$, $d\tau(P_{ij})/d\tau(P) = 1$ from Eq. (27), and all coefficients are constants. If the field is produced by a line source at Q , it follows from Eqs. (4) and (5) that $[\kappa_0(Q_{ij})]^{-\frac{1}{2}} A_i(Q_{ij})$ does not depend upon the distance from Q to Q_{ij} but depends only upon the direction $\theta_{ij}(Q)$ of the ray from Q to Q_{ij} . Calling this product $A_i'[\theta_{ij}(Q)]$, we may write Eq. (34) as

$$A_{ij}(P) = R_j' R_i' (V_i / V_j)^{\frac{1}{2}} A_i'[\theta_{ij}(Q)]. \quad (35)$$

The quantity $A_i'[\theta_{ij}(Q)]$ expresses the strength and angular pattern or radiation pattern of the source and depends upon the source characteristics. In the case of a unit force per unit length in the x direction, A_i' is given by Eqs. (15) and (16). Upon using Eqs. (21) and (22) in Eq. (12), we obtain $\mathbf{u}(P)$ due to a line source at Q in a semi-infinite homogeneous medium.

IV. DETERMINATION OF THE RADIATION COEFFICIENT

We now use the preceding results to determine the field $\mathbf{u}(P)$ due to a compressional line source at Q in a semi-infinite homogeneous medium bounded by a free surface. Then, we compare the results with the exact

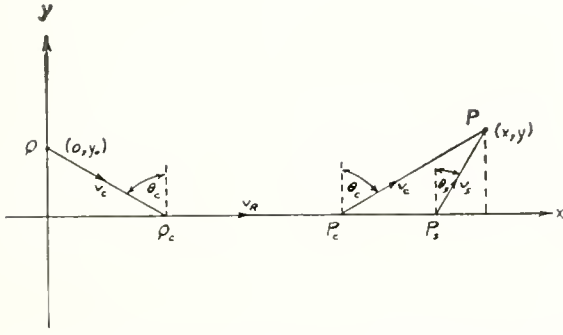


FIG. 2. An elastic medium with a free surface at $y=0$ occupies the half-space $y \geq 0$. The compressional line source is parallel to the z axis and intersects the plane $z=0$ at Q with coordinates $x=0$, $y=y_0$. There are two ray paths from the source to the observation point at P and are denoted by QQ_cP_cP and QQ_sP_sP .

solution of the equations of elasticity for this case, when that solution is expanded asymptotically for large values of the frequency ω . This comparison will show that the two results are identical when the radiation coefficients R_i' are chosen appropriately. We then assume that these coefficients have the values so determined in all cases involving a free surface, provided the material constants in the expressions for R_i' are given the values appropriate to the point of excitation or radiation.

We begin by observing that the point Q_{ij} depends only upon i and Q , while the point P_{ij} depends only upon j and P . This is because the ray from Q to Q_{ij} is a straight line that makes an angle with the normal to the surface determined by Eq. (1). In Eq. (1), only V_i occurs, so the angle depends on V_i , and, therefore, the location of Q_{ij} depends only upon V_i and the location of Q . The same considerations apply to P_{ij} . Therefore, we write Q_i instead of Q_{ij} and P_j instead of P_{ij} .

Let the medium occupy the half-space $y > 0$ with a free surface at $y=0$. Let the line source be parallel to the z axis and let it intersect the plane $z=0$ at Q with coordinates $x=0$, $y=y_0$. Then, the coordinates of Q_c are $(y_0 \tan \theta_c, 0)$. If the observation point P is at (x, y) with $x > 0$, then P_c is at $(x - y \tan \theta_c, 0)$ and P_s is at $(x - y \tan \theta_s, 0)$ (see Fig. 2). Between these points, we find the distances to be

$$\begin{aligned} d(Q, Q_c) &= y_0 \sec \theta_c; \\ d(Q_c, P_c) &= x - (y_0 + y) \tan \theta_c; \end{aligned} \quad (36)$$

$$\begin{aligned} d(P_c, P) &= y \sec \theta_c; \\ d(Q_c, P_s) &= x - y_0 \tan \theta_c - y \tan \theta_s; \\ d(P_s, P) &= y \sec \theta_s. \end{aligned} \quad (37)$$

Upon using Eqs. (36) and (37) in (33), we obtain

$$\varphi_{cc}(P) = \omega V_c^{-1} [|x| \sin \theta_c + (y_0 + y) \cos \theta_c]; \quad (38)$$

$$\varphi_{cs}(P) = \omega V_c^{-1} [|x| \sin \theta_c + y_0 \cos \theta_c + V_c^{-1} V_s y \cos \theta_s]. \quad (39)$$

Here, we have used the relations $V_s = V_R \sin \theta_s$ and $V_c = V_R \sin \theta_c$, which follow from Eq. (1). We have also

written $|x|$ instead of x , so the results will be valid for $x < 0$. Since we consider a compressional source, $A_s'(Q) = 0$, so we needn't consider the shear rays from Q .

Let us now use Eqs. (38) and (39), together with Eqs. (35) and (12), to obtain for the surface wave at P

$$\begin{aligned} u(P) &= A_c'(\theta_c) R_c'^2 \mathbf{k}_c \exp \{ i \omega V_c^{-1} [|x| \sin \theta_c \\ &\quad + (y_0 + y) \cos \theta_c] \} + A_c'(\theta_c) R_c' R_s' (V_c/V_s)^{\frac{1}{2}} \mathbf{n}_s \\ &\quad \times \exp \{ i \omega V_c^{-1} [|x| \sin \theta_c + y_0 \cos \theta_c \\ &\quad + V_c^{-1} V_s y \cos \theta_s] \}. \end{aligned} \quad (40)$$

The unit vectors \mathbf{k}_c and \mathbf{n}_s are

$$\mathbf{k}_c = (\sin \theta_c, \cos \theta_c), \quad (41)$$

$$\mathbf{n}_s = (\cos \theta_s, -\sin \theta_s). \quad (42)$$

The result (40) involves, in addition to known geometrical quantities, the three unknown quantities $A_c'(\theta_c)$, R_c' , and R_s' . Let us consider the exact solution for the displacement in an infinite homogeneous medium due to a compressional line source of unit strength (unit volume per unit length). This solution, when expanded for large values of ω , yields

$$A_c'(\theta_c) = (\omega/8\pi V_c)^{\frac{1}{2}} e^{i3\pi/4}. \quad (43)$$

Next, we consider the exact solution for the displacement due to the same source in a semi-infinite homogeneous medium with a free surface. This solution has been expanded asymptotically for large values of ω . In the notation of the previous paragraphs of this section, the surface wave at P is found to be

$$\begin{aligned} u(x, y) &\sim \mathbf{k}_c (\omega/V_c) e^{i\pi/2} (2 \sin^2 \theta_c - V_c^2 V_s^{-2})^2 [\partial G(\theta_c)/\partial \theta]^{-1} \\ &\quad \times \exp \{ i \omega V_c^{-1} [|x| \sin \theta_c + (y_0 + y) \cos \theta_c] \} \\ &\quad + \mathbf{n}_s (\omega/V_s) e^{i\pi/2} 2 \sin \theta_c (2 \sin^2 \theta_c - V_c^2 V_s^{-2}) \\ &\quad \times (V_c^2 V_s^{-2} - \sin^2 \theta_c)^{\frac{1}{2}} [\partial G(\theta_c)/\partial \theta]^{-1} \\ &\quad \times \exp \{ i \omega V_c^{-1} [|x| \sin \theta_c + y_0 \cos \theta_c \\ &\quad + V_c^{-1} V_s y \cos \theta_s] \}. \end{aligned} \quad (44)$$

In Eq. (44), the function $G(\theta)$ is defined by

$$\begin{aligned} G(\theta) &= (2 \sin^2 \theta - V_c^2 V_s^{-2})^2 + 4 \sin^2 \theta \\ &\quad \times \cos \theta (V_c^2 V_s^{-2} - \sin^2 \theta)^{\frac{1}{2}}. \end{aligned} \quad (45)$$

Upon comparing Eq. (44) with Eq. (40), we find that they agree exactly, provided that

$$A_c'(\theta_c) R_c'^2 = e^{i\pi/2} (\omega/V_c) (2 \sin^2 \theta_c - V_c^2 V_s^{-2})^2 \times [\partial G(\theta_c)/\partial \theta]^{-1}, \quad (46)$$

$$\begin{aligned} A_c'(\theta_c) R_c' R_s' (V_c/V_s)^{\frac{1}{2}} &= 2 e^{i\pi/2} (\omega/V_s) \\ &\quad \times \sin \theta_c (2 \sin^2 \theta_c - V_c^2 V_s^{-2}) (V_c^2 V_s^{-2} - \sin^2 \theta_c)^{\frac{1}{2}} \\ &\quad \times [\partial G(\theta_c)/\partial \theta]^{-1}. \end{aligned} \quad (47)$$

When Eq. (43) is used for $A_c'(\theta_c)$ in Eqs. (46) and (47), they can be solved for R_c' and R_s' . The results are

$$\begin{aligned} R_c' &= (8\pi\omega/V_c)^{\frac{1}{2}} e^{-i\pi/8} (2 \sin^2 \theta_c - V_c^2 V_s^{-2}) \\ &\quad \times [\partial G(\theta_c)/\partial \theta]^{-\frac{1}{2}}, \end{aligned} \quad (48)$$

$$R_s' = (8\pi\omega/V_c)^{1/2} e^{-i\pi/8} 2 \sin\theta_c \cos\theta_c \\ \times [\partial G(\theta_c)/\partial\theta]^{-1/2} (V_c/V_s)^{1/2}. \quad (49)$$

The above equations determine the radiation coefficients.

Let us now insert Eqs. (48), (49), and (32) into Eq. (12) and replace the amplitudes $A_c(Q_{cj})$ and $A_s(Q_{sj})$ by Eqs. (4) and (5). This yields the final result for the sur-

face wave excited on a curved free surface by an arbitrary incident field in an inhomogeneous elastic medium. It is

$$\mathbf{u}(P) = \sqrt{\pi} e^{-i(\pi/4)} \omega^{1/2} [\mathbf{B}_{cc}(P) e^{i\omega\varphi_{cc}'(P)} + \mathbf{B}_{sc}(P) e^{i\omega\varphi_{sc}'(P)} \\ + \mathbf{B}_{cs}(P) e^{i\omega\varphi_{cs}'(P)} + \mathbf{B}_{ss}(P) e^{i\omega\varphi_{ss}'(P)}], \quad (50)$$

where

$$\mathbf{B}_{cc}(P) = 2(2)^{1/2} A_c(Q_{cc}) [V_c(P_{cc})]^{-1/2} \left[\frac{V_c(Q_{cc}) \rho(Q_{cc}) d\sigma(Q_{cc}) d\tau(P_{cc})}{V_c(P) \rho(P) d\sigma(P_{cc}) d\tau(P)} \right]^{1/2} \cdot [2 \sin^2\theta_c - V_c^2 V_s^{-2}]_{P_{cc}} \\ \times [2 \sin^2\theta_c - V_c^2 V_s^{-2}]_{Q_{cc}} \left[\frac{\partial G(Q_c)}{\partial\theta} \right]_{P_{cc}}^{-1/2} \left[\frac{\partial G(\theta_c)}{\partial\theta} \right]_{Q_{cc}}^{-1/2} \cdot [\kappa_0 + (\kappa + \theta_c)]_{Q_{cc}}^{-1/2} \mathbf{k}_{cc}(P); \quad (51)$$

$$\mathbf{B}_{sc}(P) = 2(2)^{1/2} A_s(Q_{sc}) [V_c(P_{sc})]^{-1/2} \left[\frac{V_c(Q_{sc})}{V_s(Q_{sc})} \right]^{5/4} \left[\frac{V_s(Q_{sc}) \rho(Q_{sc}) d\sigma(Q_{sc}) d\tau(P_{sc})}{V_c(P) \rho(P) d\sigma(P_{sc}) d\tau(P)} \right]^{1/2} \cdot [2 \sin^2\theta_c - V_c^2 V_s^{-2}]_{P_{sc}} \\ \times [2 \sin\theta_c \cos\theta_c]_{Q_{sc}} \left[\frac{\partial G(\theta_c)}{\partial\theta} \right]_{P_{sc}}^{-1/2} \left[\frac{\partial G(\theta_c)}{\partial\theta} \right]_{Q_{sc}}^{-1/2} \cdot [\kappa_0 + (\kappa + \theta_s)]_{Q_{sc}}^{-1/2} \mathbf{k}_{sc}(P); \quad (52)$$

$$\mathbf{B}_{cs}(P) = 2(2)^{1/2} A_c(Q_{cs}) [V_s(P_{cs})]^{-1/2} \left[\frac{V_c(P_{cs})}{V_s(P_{cs})} \right]^{5/4} \left[\frac{V_c(Q_{cs}) \rho(Q_{cs}) d\sigma(Q_{cs}) d\tau(P_{cs})}{V_s(P) \rho(P) d\sigma(P_{cs}) d\tau(P)} \right]^{1/2} \cdot [2 \sin^2\theta_c - V_c^2 V_s^{-2}]_{Q_{cs}} \\ \times [2 \sin\theta_c \cos\theta_c]_{P_{cs}} \left[\frac{\partial G(\theta_c)}{\partial\theta} \right]_{Q_{cs}}^{-1/2} \left[\frac{\partial G(\theta_c)}{\partial\theta} \right]_{P_{cs}}^{-1/2} \cdot [\kappa_0 + (\kappa + \theta_c)]_{Q_{cs}}^{-1/2} \mathbf{n}_{cs}(P); \quad (53)$$

$$\mathbf{B}_{ss}(P) = 2(2)^{1/2} A_s(Q_{ss}) [V_s(P_{ss})]^{-1/2} \left[\frac{V_c(P_{ss})}{V_s(P_{ss})} \right]^{5/4} \left[\frac{V_s(Q_{ss})}{V_s(P_{ss})} \right]^{5/4} \cdot \left[\frac{V_s(Q_{ss}) \rho(Q_{ss}) d\sigma(Q_{ss}) d\tau(P_{ss})}{V_s(P) \rho(P) d\sigma(P_{ss}) d\tau(P)} \right]^{1/2} \\ \cdot [2 \sin\theta_c \cos\theta_c]_{P_{ss}} [2 \sin\theta_c \cos\theta_c]_{Q_{ss}} \cdot \left[\frac{\partial G(\theta_c)}{\partial\theta} \right]_{Q_{ss}}^{-1/2} \left[\frac{\partial G(\theta_c)}{\partial\theta} \right]_{P_{ss}}^{-1/2} \cdot [\kappa_0 + (\kappa + \theta_s)]_{Q_{ss}}^{-1/2} \mathbf{n}_{ss}(P); \quad (54)$$

and the phase $\varphi_{ij}'(P)$, which is independent of ω , is given by $\varphi_{ij}(P)/\omega$.

V. TRANSIENT WAVES

The results obtained in the preceding section apply to time-harmonic waves. We now transform them to apply to transient waves. Suppose that we have a time-harmonic solution $\mathbf{u}(\mathbf{x}, \omega) e^{-i\omega t}$ of the elastic equations of motion for a source distribution with harmonic time dependence. Then it is easily shown, by means of Fourier transforms, that the time-dependent solution $\mathbf{U}(\mathbf{x}, t)$ for the same source distribution with an arbitrary time dependence $F(t)$ is given by

$$\mathbf{U}(\mathbf{x}, t) = \int_{-\infty}^{\infty} \mathbf{u}(\mathbf{x}, \omega) f(\omega) e^{-i\omega t} d\omega. \quad (55)$$

Here, $f(\omega)$ is the Fourier transform of $F(t)$ and

$$F(t) = \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega. \quad (56)$$

It is possible to use Eq. (55) in its present form to find the waveform for transient waves. However, it is simpler to use another procedure that makes use of the special form of the frequency dependence obtained in Eq. (50). The time-harmonic solution given by Eq. (50) is of the form

$$\mathbf{u}(\mathbf{x}, t) = \sqrt{\pi} e^{-i(\pi/4)} \omega^{1/2} \sum_{i,j} \mathbf{B}_{ij} e^{i\omega\varphi_{ij}'}, \quad (57)$$

where the summation consists of four terms $i=c, s$; $j=c, s$. The amplitudes \mathbf{B}_{ij} and phases φ_{ij}' , which are independent of frequency, are easily found by comparison of Eq. (57) with (50)–(54). Using Eq. (57) in (55), we obtain

$$\mathbf{U}(\mathbf{x}, t) = \sqrt{\pi} e^{-i(\pi/4)} \sum_{i,j} \mathbf{B}_{ij} \int_{-\infty}^{\infty} \omega^{1/2} e^{-i\omega(t-\varphi_{ij}')} f(\omega) d\omega. \quad (58)$$

Now, we differentiate Eq. (56) α times with respect to t and obtain

$$\frac{d^\alpha}{dt^\alpha} F(t) = (-i)^\alpha \int_{-\infty}^{\infty} \omega^\alpha e^{-i\omega t} f(\omega) d\omega. \quad (59)$$

If we let $\alpha = \frac{1}{2}$, then a comparison of Eq. (58) with (59) shows that

$$U(x, t) = \sqrt{\pi} \sum_{i,j} B_{ij} (d/dt)^{\frac{1}{2}} F(t - \varphi_{ij}'). \quad (60)$$

Thus, the desired transient waveform for a given input $F(t)$ is given by the one-half-order derivative of $F(t - \varphi_{ij}')$. It is well-known that the one-half-order derivative is given by either of the following expressions⁸:

$$\left(\frac{d}{dt}\right)^{\frac{1}{2}} F(t) = \frac{1}{\sqrt{\pi}} \int_0^t (t - \tau)^{-\frac{1}{2}} \frac{d}{d\tau} F(\tau) d\tau. \quad (61)$$

$$\left(\frac{d}{dt}\right)^{\frac{1}{2}} F(t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t (t - \tau)^{-\frac{1}{2}} F(\tau) d\tau. \quad (62)$$

Upon substituting Eq. (61) or (62) into (60), we obtain two different expressions for the transient waveform. They are

$$U(x, t) = \sum_{i,j} B_{ij} \int_0^{t - \varphi_{ij}'} (t - \varphi_{ij}' - \tau)^{-\frac{1}{2}} \frac{d}{d\tau} F(\tau) d\tau, \quad (63)$$

⁸ R. Courant, *Differential and Integral Calculus* (Interscience Publishers, Inc., New York, 1936), Vol. 2, pp. 339-341.

and

$$U(x, t) = \sum_{i,j} B_{ij} \frac{d}{dt^*} \int_0^{t^*} (t^* - \tau)^{-\frac{1}{2}} F(\tau) d\tau, \quad (64)$$

where $t^* = t - \varphi_{ij}'$ is the delayed time. The above equations yield the transient response for an arbitrary pulse $F(t)$ and are the main results of this section.

As a simple example, suppose that $F(t)$ is the unit step function:

$$F(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}. \quad (65)$$

Then, $dF(t)/dt = \delta(t)$, and Eq. (63) yields immediately

$$U(x, t) = \sum_{i,j} B_{ij} (t - \varphi_{ij}')^{-\frac{1}{2}}. \quad (66)$$

Other pulses may be treated in a similar manner.

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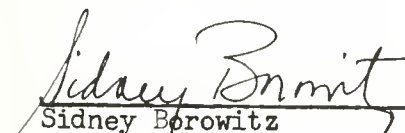
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HOW DARK IS THE SHADOW OF A ROUND-ENDED SCREEN?

Joseph B. Keller


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Acting Director


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Contract Monitor

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Abstract

The amplitudes of the fields in the shadows of two screens with rounded ends are determined as functions of b/λ . Here b denotes the radius of curvature of the end of the screen and λ denotes the wavelength of the normally incident plane wave. For one of the screens the amplitude decreases monotonically as b/λ increases when the field vanishes on the screen. But if the normal derivative vanishes the amplitude first oscillates as b/λ increases and then finally decreases. For the other screen - the more realistic one - the amplitude probably decreases monotonically in both cases as b/λ increases.

The calculations are based on the geometrical theory of diffraction. For the second screen a calculation from the exact series solution is also made. The close agreement between the two results is a confirmation of the geometrical diffraction theory.

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1. Introduction

Because of diffraction, there is always some illumination in a shadow. The intensity of this illumination depends upon the wavelength λ of the incident radiation and upon the size and shape of the shadow-forming object. In order to examine this dependence, we have considered the shadows cast by two semi-infinite screens with rounded tips of radius b upon either of which a plane wave of unit amplitude is normally incident. (See Figures 1 and 2). We find that in either case, at a large distance r from the tip the diffracted field u in the shadow has the form

$$(1) \quad u = \frac{e^{i(kr + \frac{\pi}{4})}}{\sqrt{kr}} f(\theta, kb) .$$

In (1) $k = 2\pi/\lambda$ is the propagation constant and $f(\theta, kb)$ is the diffraction pattern or complex amplitude of the cylindrical wave scattered from the tip in the direction θ . The angle θ is measured from the shadow boundary.

In Figure 3, the amplitude $|f(\theta, kb)|$ is shown as a function of kb for $\theta = \pi/4$ for the screen in Figure 2. The encircled points show that the shadow cast by this screen becomes darker as kb increases when $u = 0$ on the screen. Thus, the shadow becomes darker as the radius of the tip increases as compared to the wavelength. The factor $k^{-1/2}$ in the denominator of (1) also makes the shadow become darker as the wavelength decreases, independently of the radius of the tip. On the other hand, if $\partial u / \partial n = 0$ then $|f(\frac{\pi}{4}, kb)|$ oscillates as kb increases. Ultimately it ceases oscillating and decays monotonically as (6) shows. This oscillation is due to interference between the two diffracted rays which occur in this case. Although the same two rays occur when $u = 0$ on the screen, one of them decays so rapidly that it produces negligible interference and no oscillation.

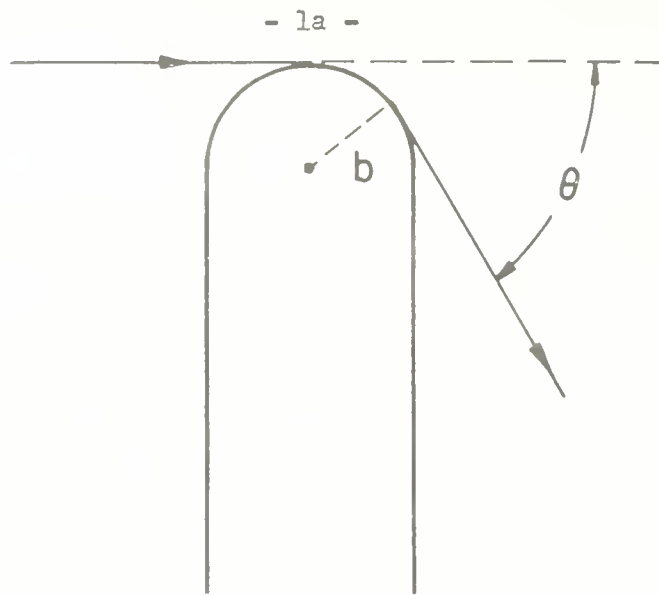


Figure 1 Cross-section of a screen of width $2b$ with a rounded tip of radius b . A plane wave is normally incident upon it from the left. The dotted line is the shadow boundary. An incident ray which grazes the tip is shown together with one of the diffracted rays which it produces in the shadow. The angle between the diffracted ray and the shadow boundary is θ .

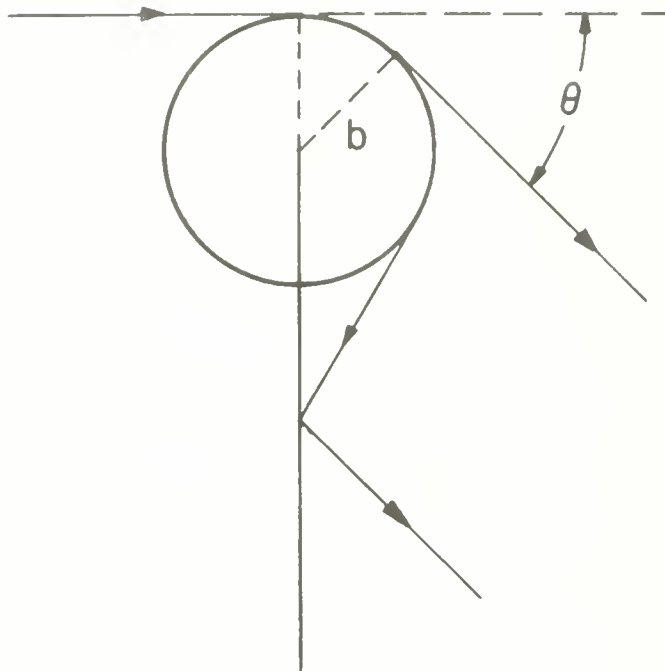


Figure 2 Cross-section of an infinitely thin screen with a cylindrical tip of radius b . A plane wave is normally incident upon it from the left. The dotted line is the shadow boundary. An incident ray which grazes the tip is shown together with the two diffracted rays which it produces making the angle θ with the shadow boundary.

An important feature of Figure 3 is the close agreement between the predictions of the geometrical theory of diffraction and the exact solutions of the boundary value problems for the field.

In Figure 4 the amplitude $|f(\theta, kb)|$ is shown as a function of kb for $\theta = \pi/4$ for the screen of Figure 1. From the physical significance of the approximations on which this figure is based, it is likely that only the decreasing parts of the curves are correct. For this screen the amplitude probably decreases monotonically for all values of kb .

2. Analysis

If the radius of the tip is zero either screen is a half-plane. In this case Sommerfeld's well-known expression for u is of the form (1) with f given by

$$(2) \quad f(\theta, 0) = \frac{1}{2\sqrt{2\pi}} \left[\sec \frac{1}{2}(\theta + \pi) \pm \csc \frac{1}{2}(\theta + \pi) \right].$$

The upper sign in (2) applies to a screen on which $u = 0$, while the lower sign applies to one on which $\partial u / \partial n = 0$. At the other extreme, when kb is large, the field in the shadow can be determined by means of the geometrical theory of diffraction [1]. In Figures 1 and 2 are shown the one or two diffracted rays which go in a given direction. By the methods and formulas in [1] we find that far from the tip the field associated with the ray in Figure 1 is given by

$$(3) \quad u_1 = \frac{e^{i(kr + \frac{\pi}{4})}}{\sqrt{kr}} \cdot (kb)^{1/3} \frac{C_0}{\sqrt{2}} e^{i\gamma_0(kb)^{1/3}\theta} e^{i(kb\theta + \frac{\pi}{12} - \frac{\pi}{4})}.$$

This same expression applies to the upper ray in Figure 2, while for the lower ray we find

$$(4) \quad u_2 = \mp \frac{e^{i(kr + \frac{\pi}{4})}}{\sqrt{kr}} \cdot (kb)^{1/3} \frac{C_0}{\sqrt{2}} e^{i\gamma_0(kb)^{1/3}(\pi - \theta)} e^{i(kb|\pi - \theta| + \frac{\pi}{12} - \frac{\pi}{4})}.$$

The upper sign in (4) applies if $u = 0$ on the screen, while the lower sign applies if $\partial u / \partial n = 0$. In these two cases the constants C_0 and γ_0 are given by

$u = 0$	$\frac{\partial u}{\partial n} = 0$
$C_0 \quad .910719$	1.53228
$\gamma_0 \quad 1.855757 e^{i\pi/3}$	$.808617 e^{i\pi/3}$

The fields $u = u_1$ and $u = u_1 + u_2$ are of the form (1) with f given by the respective formulas:

$$(5) \quad f(\theta, kb) = (kb)^{1/3} \frac{C_0}{\sqrt{2}} e^{-i\pi/6} e^{i[kb + \gamma_0 (kb)^{1/3}]} e$$

$$(6) \quad f(\theta, kb) = (kb)^{1/3} \frac{C_0}{\sqrt{2}} e^{-i\pi/6} \left[e^{i[kb + \gamma_0 (kb)^{1/3}]} + e^{i[kb + \gamma_0 (kb)^{1/3}](\pi - \theta)} \right]$$

In Figures 3 and 4 the values of $|f(\frac{\pi}{4}, kb)|$ are shown based on (2) for $kb = 0$ and on (5) and (6) (solid line) for $kb \geq .1$ for screens on which either $u = 0$ (lower graph) or $\partial u / \partial n = 0$ (upper graph).

For the screen of Figure 2 the field u has been determined exactly for all values of kb by the method of separation of variables [2]. The result for f , which coincides with (2) for $kb = 0$, has been expanded asymptotically for kb large [3]. It then agrees with (6). We have used this solution to calculate $|f|$ in the range $0 < kb \leq 10$. The results are shown as circled points in Figure 3 for both the case in which $u = 0$ (lower graph) and that in which $\partial u / \partial n = 0$ (upper graph) on the screen. The broken line in the upper graph of Figure 3 for the case $\partial u / \partial n = 0$ is based upon (6) with the factor $kb + \gamma_0 (kb)^{1/3}$ in the exponents replaced by the more accurate expression $kb + \gamma_0 (kb)^{1/3} - \sigma_0 (kb)^{-1/3}$. The constant σ_0 has the value [4] $\sigma_0 = .145463 e^{-i\pi/3}$.

[2]

The agreement between the circled points based upon the series solution and the simple results (6) of the geometrical theory of diffraction [1] is

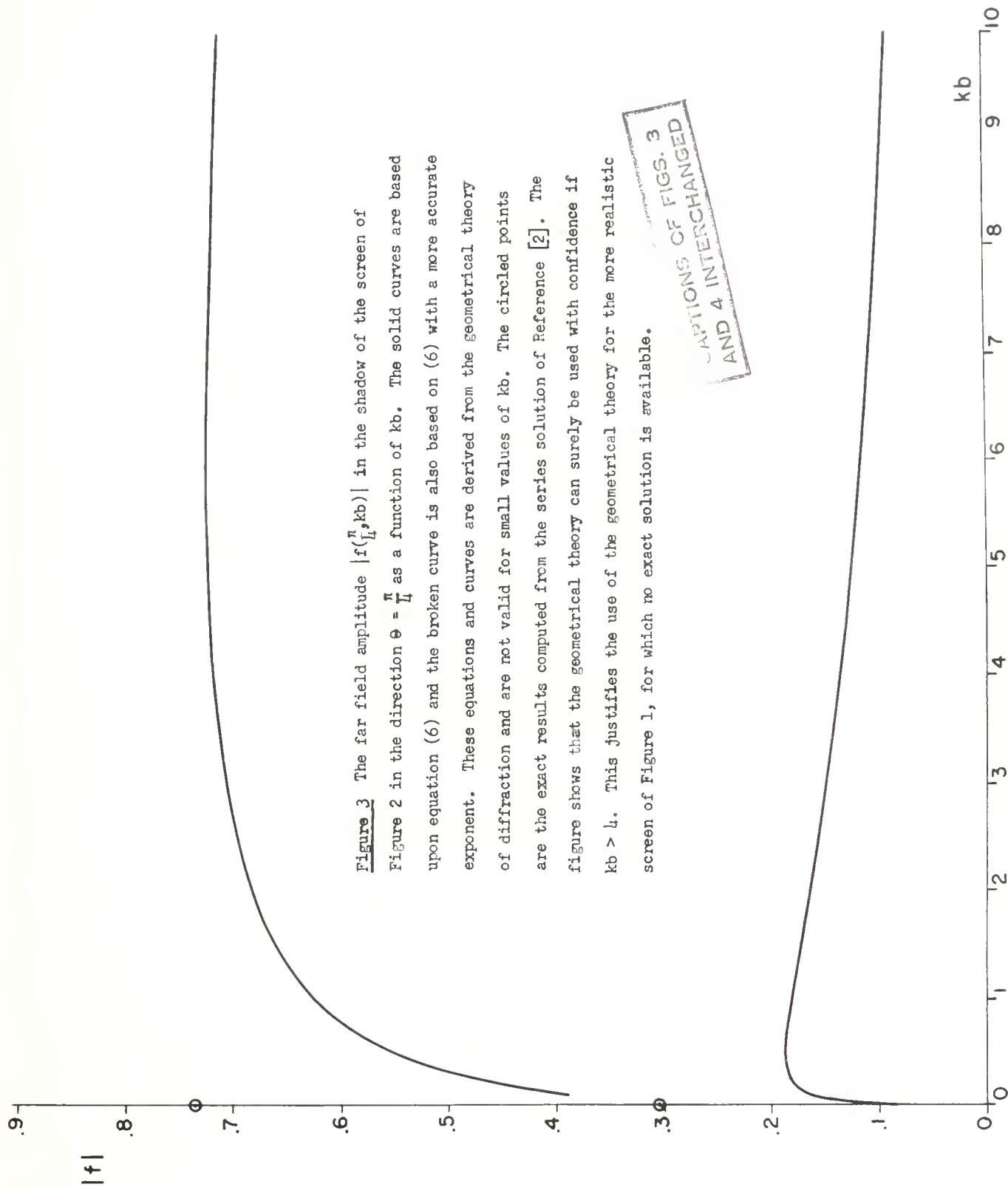


Figure 3 The far field amplitude $|f(\frac{\pi}{L}, kb)|$ in the shadow of the screen of Figure 2 in the direction $\theta = \frac{\pi}{L}$ as a function of kb . The solid curves are based upon equation (6) and the broken curve is also based on (6) with a more accurate exponent. These equations and curves are derived from the geometrical theory of diffraction and are not valid for small values of kb . The circled points are the exact results computed from the series solution of Reference [2]. The figure shows that the geometrical theory can surely be used with confidence if $kb > 4$. This justifies the use of the geometrical theory for the more realistic screen of Figure 1, for which no exact solution is available.

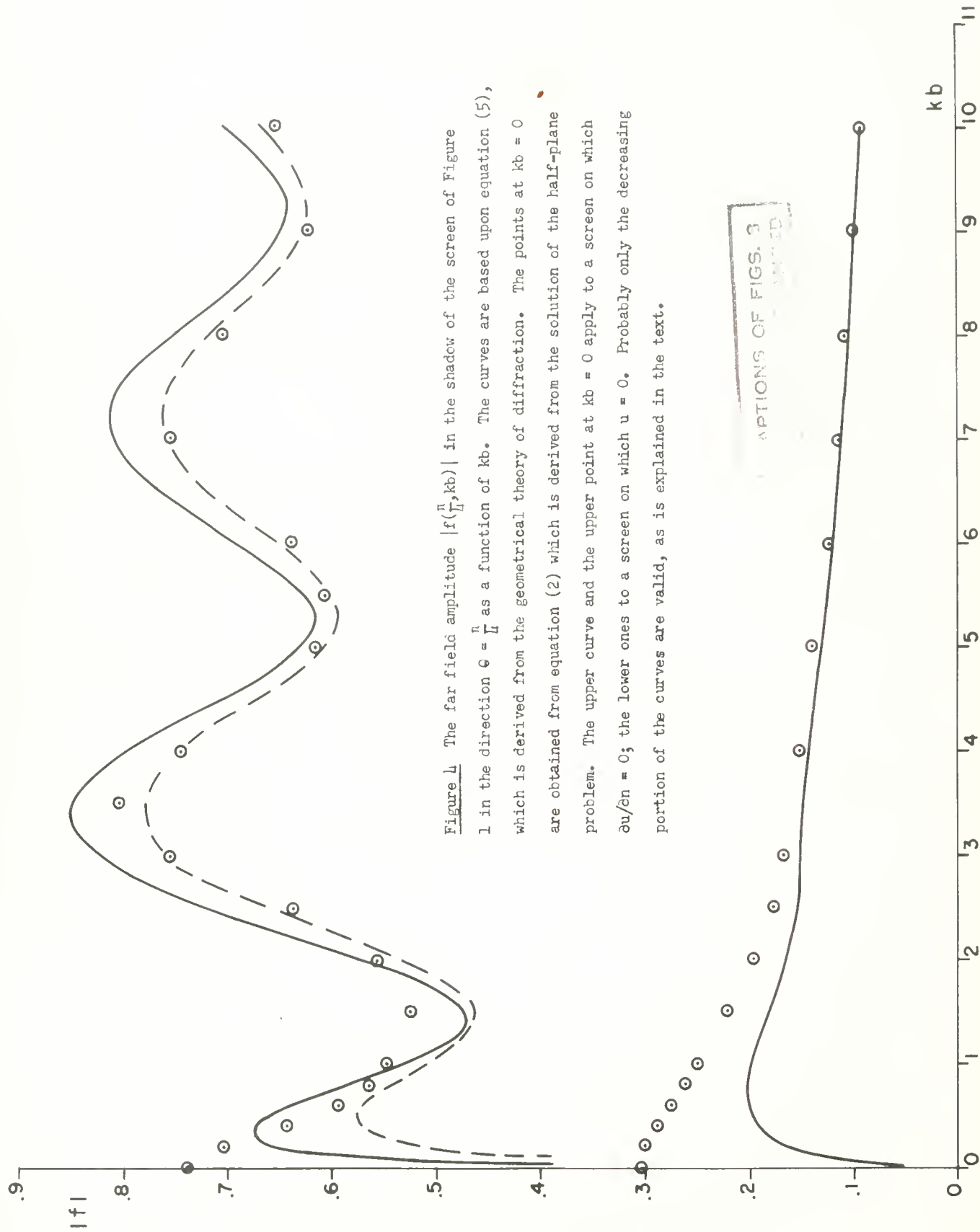


Figure 1 The far field amplitude $|f(\frac{\pi}{L}, kb)|$ in the shadow of the screen of Figure 1 in the direction $\theta = \frac{\pi}{L}$ as a function of kb . The curves are based upon equation (5), which is derived from the geometrical theory of diffraction. The points at $kb = 0$ are obtained from equation (2) which is derived from the solution of the half-plane problem. The upper curve and the upper point at $kb = 0$ apply to a screen on which $\partial u / \partial n = 0$; the lower ones to a screen on which $u = 0$. Probably only the decreasing portion of the curves are valid, as is explained in the text.

ADDITIONS OF FIGS. 3
AND 4



another confirmation of the geometrical theory. The labor involved in computing the curves from (6) was almost negligible. However, the calculations of the points from the series solution required a great deal of time. Four series of Bessel functions had to be computed for each point and each required about $1.2 kb + 2$ terms to yield three decimal accuracy. To use the series for $kb > 10$ would require tables of Bessel functions of high order as well as many terms. In view of the agreement of the series result with (6), this does not seem to be necessary. Thus, for example, we may conclude from (6) that for kb large $|f|$ ultimately decreases.

Acknowledgment

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